BELL POLYNOMIALS AND k-GENERALIZED DYCK PATHS

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Abstract. A *k-generalized Dyck path* of length n is a lattice path from $(0,0)$ to $(n,0)$ in the plane integer lattice $\mathbb{Z}\times\mathbb{Z}$ consisting of horizontal-steps $(k,0)$ for a given integer $k\geq 0$, up-steps $(1,1)$, and down-steps $(1,-1)$, which never passes below the x-axis. The present paper studies three kinds of statistics on k-generalized Dyck paths: "number of u-segments", "number of internal u-segments" and "number of (u,h)-segments". The Lagrange inversion formula is used to represent the generating function for the number of k-generalized Dyck paths according to the statistics as a sum of the partial Bell polynomials or the potential polynomials. Many important special cases are considered leading to several surprising observations. Moreover, enumeration results related to u-segments and (u,h)-segments are also established, which produce many new combinatorial identities, and specially, two new expressions for Catalan numbers.

Keywords: Bell polynomials, Potential polynomials, k-paths, Catalan numbers

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1. Introduction

Let $\mathfrak{L}_{n,k}$ denote the set of lattice paths of length n from $(0,0)$ to $(n,0)$ in the plane integer lattice $\mathbb{Z}\times\mathbb{Z}$ consisting of horizontal-steps $h=(k,0)$ for a given integer $k\geq 0$, up-steps $u=(1,1)$, and down-steps $d=(1,-1)$. Let $\mathfrak{L}_{n,k}^{m,j}$ be the set of lattice paths in $\mathfrak{L}_{n,k}$ with m up-steps and j horizontal-steps. Let L be any lattice path in $\mathfrak{L}_{n,k}^{m,j}$. A *u-segment* of L is a maximal sequence of consecutive up-steps in L. Define $\alpha_i(L)$ to be the number of u-segments of length i in L and call L having the u-segments of type $1^{\alpha_1(L)}2^{\alpha_2(L)}\cdots$. Let $\mathfrak{L}_{n,k,r}^{m,j}$ be the subset of lattice paths in $\mathfrak{L}_{n,k}^{m,j}$ with r u-segments.

A *k-generalized Dyck path* or *k-path* (for short) of length n is a lattice path in $\mathfrak{L}_{n,k}$ which never passes below the x-axis. By our notation, a *Dyck path* is a 0-path, a *Motzkin path* is a 1-path and a *Schröder path* is a 2-path. Let $\mathfrak{P}_{n,k}^{m,j}$ denote the set of k-paths of length n (i.e. $n=2m+kj$) with m up-steps and j horizontal-steps and let $\mathfrak{Q}_{n,k}^{m,j}$ be the subset of k-paths in $\mathfrak{P}_{n,k}^{m,j}$ with no horizontal-step at x-axis. Define $\mathfrak{P}_{n,k,r}^{m,j}$ ($\mathfrak{Q}_{n,k,r}^{m,j}$) to be the subset of k-paths in $\mathfrak{P}_{n,k}^{m,j}$ ($\mathfrak{Q}_{n,k}^{m,j}$) with r u-segments.

In [9], we study two kinds of statistics on Dyck paths: "number of u-segments" and "number of internal u-segments". In this paper, we consider these two statistics together with "number of (u,h)-segments" in the more extensive setting of k-paths. In order to do this, we present two necessary tools : Lagrange inversion formula and the potential polynomials.

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Lagrange Inversion Formula [10]**.** If $f(x) = \sum_{n\geq 1} f_n x^n$ with $f_1 \neq 0$, then the coefficients of the composition inverse $g(x)$ of $f(x)$ (namely, $f(g(x)) = g(f(x)) = x$) can be given by

$$[x^n]g(x) = \frac{1}{n}[x^{n-1}]\Big(\frac{x}{f(x)}\Big)^n. \tag{1.1}$$

More generally, for any formal power series $\Phi(x)$,

$$[x^n]\Phi(g(x)) = \frac{1}{n}[x^{n-1}]\Phi'(x)\Big(\frac{x}{f(x)}\Big)^n, \tag{1.2}$$

for all $n \geq 1$, where $\Phi'(x)$ is the derivative of $\Phi(x)$ with respect to x.

The Potential Polynomials [5, pp. 141,157]**.** The potential polynomials $\mathbf{P}_n^{(\lambda)}$ are defined for each complex number λ by

$$1 + \sum_{n\geq 1} \mathbf{P}_n^{(\lambda)} \frac{x^n}{n!} \quad = \quad \Big\{1 + \sum_{n\geq 1} f_n \frac{x^n}{n!}\Big\}^\lambda,$$

which can be represented by Bell polynomials

$$\mathbf{P}_n^{(\lambda)} = \mathbf{P}_n^{(\lambda)}(f_1, f_2, f_3, \dots) = \sum_{1\leq k\leq n} \binom{\lambda}{k} k! \mathbf{B}_{n,k}(f_1, f_2, f_3, \dots), \tag{1.3}$$

where $\mathbf{B}_{n,i}\big(x_1, x_2, \cdots\big)$ is the partial Bell polynomial on the variables $\{x_j\}_{j\geq 1}$ (see [2]).

In this paper, with the Lagrange inversion formula, we can represent the generating functions for the number of k-paths according to our statistics (see Sections 2-4) as a sum of partial Bell polynomials or the potential polynomials. For example,

$$\sum_{P\in\mathfrak{P}_{n,k}^{m,j}} \prod_{i\geq 1} t_i^{\alpha_i(P)} \quad = \quad \frac{1}{m+1}\binom{m+j}{m} \sum_{r=0}^{m} \binom{m+j+1}{r} \frac{r!}{m!} \mathbf{B}_{m,r}\big(1!t_1, 2!t_2, \cdots\big),$$

$$\sum_{Q\in\mathfrak{Q}_{n,k}^{m,j}} \prod_{i\geq 1} t_i^{\alpha_i(Q)} \quad = \quad \frac{m}{(m+j+1)(m+j)} \binom{m+j}{j} \frac{\mathbf{P}_m^{(m+j+1)}(1!t_1, 2!t_2, \cdots)}{m!}.$$

We consider a number of important special cases. These lead to several surprising results. Moreover, enumeration results related to u-segments and (u,h)-segments are also established in Section 5, producing many new combinatorial identities and in particular the following two new expressions for the Catalan numbers:

$$\sum_{p=0}^{[n/2]} \frac{1}{2p+1}\binom{3p}{p}\binom{n+p}{3p} \quad = \quad \frac{1}{n+1}\binom{2n}{n}, \qquad (n \geq 0)$$

$$\sum_{p=0}^{[(n-1)/2]} \frac{1}{2p+1}\binom{3p+1}{p+1}\binom{n+p}{3p+1} \quad = \quad \frac{1}{n+1}\binom{2n}{n}, \qquad (n \geq 1).$$

2. "u-SEGMENTS" STATISTICS IN k-PATHS

We start this section by studying the generating function for the number of k-paths of length n according to the statistics $\alpha_1, \alpha_2, \ldots$, that is,

$$P(x, z; \mathbf{t}) = P(x, z; t_1, t_2, \dots) = \sum_{m,j\geq 0} x^m z^j \sum_{P\in\mathfrak{P}_{n,k}^{m,j}} \prod_{i\geq 1} t_i^{\alpha_i(P)}.$$

Proposition 2.1. *The ordinary generating function $P(x,z;\mathbf{t})$ is given by*

$$P(x,z;\mathbf{t}) = 1 + zP(x,z;\mathbf{t}) + \sum_{j\geq 1} t_j x^j P^j(x,z;\mathbf{t}) + z\sum_{j\geq 1} t_j x^j P^{j+1}(x,z;\mathbf{t}). \tag{2.1}$$

Proof. Note that $P(x,z;\mathbf{t})$ can be written as $P(x,z;\mathbf{t}) = 1 + zP(x,z;\mathbf{t}) + \sum_{j\geq 1} P_j(x,z;\mathbf{t})$, where $P_j(x,z;\mathbf{t})$ is the generating function for the number of k-paths with initial u-segment of length j according to the statistics $\alpha_1, \alpha_2, \ldots$. An equation for $P_j(x,z;\mathbf{t})$ is obtained from the first return decomposition of a k-path starting with a u-segment of length j: either $P = u^j dP^{(1)} dP^{(2)} d \ldots P^{(j-1)} dP^{(j)}$ or $P = u^j hP^{(1)} dP^{(2)} d \ldots P^j dP^{(j+1)}$, where $P^{(1)}, \ldots, P^{(j)}$ are k-paths, see Figure 1.

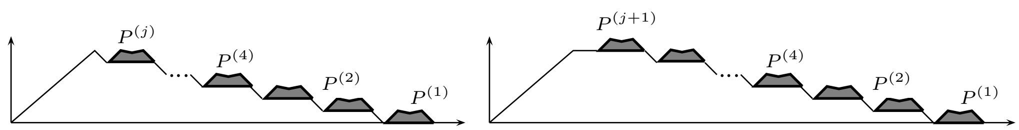

FIGURE 1. First return decomposition of a k-path starting with exactly j up-steps.

Thus $P_j(x,z;\mathbf{t}) = t_j x^j P^j(x,z;\mathbf{t}) + zt_j x^j P^{j+1}(x,z;\mathbf{t})$. Hence, $P(x,z;\mathbf{t})$ satisfies $P(x,z;\mathbf{t}) = 1 + zP(x,z;\mathbf{t}) + \sum_{j\geq 1} t_j x^j P^j(x,z;\mathbf{t}) + z\sum_{j\geq 1} t_j x^j P^{j+1}(x,z;\mathbf{t})$, as required. $\square$

Define $y = y(x,z;\mathbf{t}) = xP(x,z;\mathbf{t})$ and $T(x) = 1 + \sum_{j\geq 1} t_j x^j$. Then (2.1) reduces to $y = (x+zy)T(y)$. Let $y^* = y(x,zx;\mathbf{t})$, then we have $y^* = x(1+zy^*)T(y^*)$.

Theorem 2.2. *For any integers $n, m \geq 1$ and $k, j \geq 0$,*

$$\sum_{P\in\mathfrak{P}_{n,k}^{m,j}} \prod_{i\geq 1} t_i^{\alpha_i(P)} = \frac{1}{m+1}\binom{m+j}{m}\sum_{r=0}^{m}\binom{m+j+1}{r}\frac{r!}{m!}\mathbf{B}_{m,r}(1!t_1, 2!t_2, \cdots).$$

Proof. Using (1.2) and (1.3), we obtain

$$\begin{aligned}
\sum_{P\in\mathfrak{P}_{n,k}^{m,j}} \prod_{i\geq 1} t_i^{\alpha_i(P)} &= [x^{m+j+1}z^j]y^* = \frac{[x^{m+j}z^j]}{m+j+1}\{(1+zx)T(x)\}^{m+j+1} \\
&= \frac{1}{m+j+1}\binom{m+j+1}{j}[x^m]T(x)^{m+j+1} \\
&= \frac{1}{m+j+1}\binom{m+j+1}{j}\frac{\mathbf{P}_m^{(m+j+1)}(1!t_1, 2!t_2, \cdots)}{m!} \\
&= \frac{1}{m+1}\binom{m+j}{m}\sum_{r=0}^{m}\binom{m+j+1}{r}\frac{r!}{m!}\mathbf{B}_{m,r}(1!t_1, 2!t_2, \cdots),
\end{aligned}$$

as claimed. $\square$

Replace t_i by qt_i in Theorem 2.2 and note that

$$\mathbf{B}_{m,r}(t_1, t_2, \cdots) = \sum_{\kappa_m(r)} \frac{m!}{r_1! r_2! \cdots r_m!}\left(\frac{t_1}{1!}\right)^{r_1}\left(\frac{t_2}{2!}\right)^{r_2}\cdots\left(\frac{t_m}{m!}\right)^{r_m}, \tag{2.2}$$

$$\mathbf{B}_{m,r}(1!qt_1, 2!qt_2, \cdots) = q^r \mathbf{B}_{m,r}(1!t_1, 2!t_2, \cdots), \tag{2.3}$$

where the summation $\kappa_m(r)$ is for all the nonnegative integer solutions of $r_1 + r_2 + \cdots + r_m = r$ and $r_1 + 2r_2 + \cdots + mr_m = m$, we have

Corollary 2.3. *For any integers $n, m, r \geq 1$ and $k, j \geq 0$, there holds*

$$\sum_{P\in\mathfrak{P}_{n,k,r}^{m,j}}\prod_{i\geq 1} t_i^{\alpha_i(P)} = \frac{1}{m+1}\binom{m+j}{m}\binom{m+j+1}{r}\frac{r!}{m!}\mathbf{B}_{m,r}\big(1!t_1, 2!t_2, \cdots\big). \tag{2.4}$$

By comparing the coefficient of $t_1^{r_1}t_2^{r_2}\cdots t_m^{r_m}$ in Corollary 2.3, one can obtain that

Corollary 2.4. *The number of k-paths in $\mathfrak{P}_{n,k,r}^{m,j}$ with u-segments of type $1^{r_1}2^{r_2}\cdots m^{r_m}$ is $\frac{1}{m+1}\binom{m+j}{m}\binom{m+j+1}{r}\binom{r}{r_1,r_2,\cdots,r_m}$. Specially, the number of Dyck paths of length $2m$ with u-segments of type $1^{r_1}2^{r_2}\cdots m^{r_m}$ is $\frac{1}{m+1}\binom{m+1}{r}\binom{r}{r_1,r_2,\cdots,r_m}$, (the case $k=0$ implies $j=0$).*

2.1. **Applications.** In what follows we consider many special cases of $T(x)$. These produce several interesting results, as described in Examples 2.5-2.14. We also obtain several identities involving Stirling numbers of the first (second) kind, idempotent numbers and other combinatorial sequences.

Example 2.5. *Let $T(x) = 1 + q(e^x - 1) = (1-q) + qe^x$, that is, $t_i = q/i!$ for all $i \geq 1$. Note that $\mathbf{B}_{n,i}\big(q,q,q,\cdots\big) = S(n,i)q^i$ [5, pp.135], where $S(n,i)$ is the Stirling number of the second kind. Then Theorem 2.2 gives*

$$\begin{aligned}\sum_{P\in\mathfrak{P}_{n,k}^{m,j}}\frac{m!\prod_{i\geq 1} q^{\alpha_i(P)}}{\prod_{i\geq 1}(i!)^{\alpha_i(P)}} &= \frac{1}{m+j+1}\binom{m+j+1}{j}\sum_{i=0}^{m}\binom{m+j+1}{i}i!S(m,i)q^i \\ &= \frac{1}{m+j+1}\binom{m+j+1}{j}\sum_{i=0}^{m}\binom{m+j+1}{i}i^m q^i(1-q)^{m+j-i+1},\end{aligned}$$

which leads to $\sum_{P\in\mathfrak{P}_{n,k}^{m,j}}\frac{m!}{\prod_{i\geq1}(i!)^{\alpha_i(P)}} = \binom{m+j+1}{j}(m+j+1)^{m-1}$ when $q=1$. By Corollary 2.3, we have

$$\sum_{P\in\mathfrak{P}_{n,k,r}^{m,j}}\frac{m!}{\prod_{i\geq 1}(i!)^{\alpha_i(P)}} = \frac{1}{m+j+1}\binom{m+j+1}{j}\binom{m+j+1}{r}r!S(m,r).$$

Example 2.6. *Let $T(x) = 1 + qxe^x$ which is equivalent to $t_i = q/(i-1)!$ for all $i \geq 1$. Note that $\mathbf{B}_{m,i}\big(q, 2q, 3q, \cdots\big) = \binom{m}{i}i^{m-i}q^i$, which are called the idempotent numbers [5, pp.135] when $q = 1$. Then Corollary 2.3 leads to*

$$\sum_{P\in\mathfrak{P}_{n,k,r}^{m,j}}\prod_{i\geq 1}\left\{\frac{1}{(i-1)!}\right\}^{\alpha_i(P)} = \frac{1}{m+j+1}\binom{m+j+1}{j}\binom{m+j+1}{r}\frac{r^{m-r}}{(m-r)!}.$$

Example 2.7. *If $T(x) = (e^x-1)/x$, then $t_i = 1/(i+1)!$ for all $i \geq 1$. It is well known that the Stirling numbers of the second kind satisfy $\big(\frac{e^x-1}{x}\big)^k/k! = \sum_{m\geq 0} S(m+k,k)x^m/(m+k)!$. Thus, Theorem 2.2 leads to*

$$\sum_{P\in\mathfrak{P}_{n,k}^{m,j}}\prod_{i\geq 1}\frac{1}{\big((i+1)!\big)^{\alpha_i(P)}} = \frac{(m+j)!}{(2m+j+1)!}\binom{m+j+1}{j}S(2m+j+1, m+j+1).$$

Example 2.8. *If $T(x) = \frac{1}{x}\ln\frac{1}{1-x}$, then $t_i = 1/(i+1)$ for all $i \geq 1$. It is well known that the Stirling numbers of the first kind $s(n,i)$ satisfy $\big(\frac{1}{x}\ln\frac{1}{1-x}\big)^k/k! = \sum_{m\geq 0}|s(m+k,k)|x^m/(m+k)!$. Thus, Theorem 2.2 leads to*

$$\sum_{P\in\mathfrak{P}_{n,k}^{m,j}}\prod_{i\geq 1}\frac{1}{(i+1)^{\alpha_i(P)}} = \frac{(m+j)!}{(2m+j+1)!}\binom{m+j+1}{j}|s(2m+j+1, m+j+1)|.$$

Example 2.9. *If $T(x) = 1 + q\ln\frac{1}{1-x}$, then $t_i = q/i$ for all $i \geq 1$. Using the fact that $\mathbf{B}_{n,i}\big(0!q, 1!q, 2!q, \cdots\big) = |s(n,i)|q^i$ [5, pp.135] together with Corollary 2.3, we have*

$$\sum_{P\in\mathfrak{P}_{n,k,r}^{m,j}} \prod_{i\geq 1}\left\{\frac{1}{i}\right\}^{\alpha_i(P)} = \frac{1}{m+1}\binom{m+j}{m}\binom{m+j+1}{r}\frac{r!}{m!}|s(m,r)|.$$

Example 2.10. *If $T(x) = 1/(1-x)^{\lambda}$, then $t_i = \binom{\lambda+i-1}{i}$ for all $i \geq 1$, where λ is an indeterminant. So, Theorem 2.2 leads to*

$$\sum_{P\in\mathfrak{P}_{n,k}^{m,j}} \prod_{i\geq 1}\binom{\lambda+i-1}{i}^{\alpha_i(P)} = \frac{1}{m+1}\binom{m+j}{m}\binom{\lambda(m+j+1)+m-1}{m},$$

which generates that when $\lambda = 1$ the set $\mathfrak{P}_{n,k}^{m,j}$ is counted by $\frac{1}{m+1}\binom{m+j}{m}\binom{2m+j}{m}$, in particular, $\mathfrak{P}_{n,k}^{m,m}$ is counted by $\frac{1}{m+1}\binom{2m}{m}\binom{3m}{m}$.

Example 2.11. *Let $T(x) = 1 + x + x^2 + \cdots + x^r$, that is, $t_i = 1$ for $1 \leq i \leq r$ and $t_i = 0$ for all $i \geq r+1$. Then Theorem 2.2 gives*

$$\sum_{P\in\mathfrak{P}_{n,k}^{m,j}} \prod_{i\geq 1}^{r} 1^{\alpha_i(P)} \prod_{i\geq r+1} 0^{\alpha_i(P)} = \frac{1}{m+1}\binom{m+j}{m}\sum_{i=0}^{m+1}(-1)^i\binom{m+j+1}{i}\binom{2m+j-(r+1)i}{m+j},$$

which implies that the number of k-paths P of length $2n$ with no u-segments of length greater than r is given by

$$\frac{1}{m+1}\binom{m+j}{m}\sum_{i=0}^{m+1}(-1)^i\binom{m+j+1}{i}\binom{2m+j-(r+1)i}{m+j}.$$

Example 2.12. *Let $T(x) = \dfrac{1}{1-x} + (q-1)x^r$, that is, $t_i = 1$ for all $i \geq 1$ except for $i = r$ and $t_r = q$. Then Theorem 2.2 gives*

$$\sum_{P\in\mathfrak{P}_{n,k}^{m,j}} q^{\alpha_r(P)} = \frac{1}{m+1}\binom{m+j}{m}\sum_{i=0}^{m}\binom{m+j+1}{i}\binom{2m+j-(r+1)i}{m+j-i}(q-1)^i,$$

which implies that the number of k-paths P of length n with exactly p u-segments of length r (namely $\alpha_r(P) = p$) is given by $\frac{1}{m+1}\binom{m+j}{m}\sum_{i=0}^{m}(-1)^{i-p}\binom{m+j+1}{i}\binom{2m+j-(r+1)i}{m+j-i}\binom{i}{p}$.

Example 2.13. *If $T(x) = 1 + \dfrac{qx^r}{1-x^r} = \dfrac{1+(q-1)x^r}{1-x^r}$, then $t_i = q$ if $i \equiv 0$ mod r and 0 otherwise. Thus, Theorem 2.2 leads to*

$$\begin{aligned}\sum_{P\in\mathfrak{P}_{2rm+kj,k}^{rm,j}} \prod_{i\geq 1,\ i\equiv 0 \bmod r} q^{\alpha_i(P)} &= \tfrac{1}{rm+1}\tbinom{rm+j}{rm}\sum_{i=1}^{m}\tbinom{rm+j+1}{i}\tbinom{m-1}{m-i}q^i \\ &= \tfrac{1}{rm+1}\tbinom{rm+j}{rm}\sum_{i=0}^{m}\tbinom{rm+j+1}{i}\tbinom{(r+1)m+j-i}{m-i}(q-1)^i.\end{aligned}$$

which produces the following results. The number of k-paths in $\mathfrak{P}_{2rm+kj,k}^{rm,j}$ such that the length of any u-segment is a multiple of r (i.e., the case q=1) is given by $\frac{1}{rm+1}\binom{rm+j}{rm}\binom{(r+1)m+j}{m}$ (by Vandermonde convolution). More precisely, the number of k-paths in $\mathfrak{P}_{2rm+kj,k}^{rm,j}$ with exactly i u-segments such that the length of any u-segment is a multiple of k is given by

$$\tfrac{1}{rm+1}\tbinom{rm+j}{rm}\tbinom{rm+j+1}{i}\tbinom{m-1}{m-i} = \tfrac{1}{rm+1}\tbinom{rm+j}{rm}\sum_{p=0}^{m}(-1)^{p-i}\tbinom{rm+j+1}{p}\tbinom{(r+1)m+j-p}{m-p}\tbinom{p}{i}.$$

Example 2.14. *Let $T(x)$ be the generating function $f^r(x)$, where $f(x)$ is the generating function for complete p-ary plane trees (see, for instance,* [4, 8] *and* [7, pp.112-113]*), which satisfies the relation $f(x) = 1 + xf^p(x)$. By the Lagrange inversion formula (1.2), we can deduce $t_i = \frac{r}{pi+r}\binom{pi+r}{i}$. Then Theorem 2.2 leads to*

$$\sum_{P\in\mathfrak{P}_{n,k}^{m,j}}\prod_{i\geq 1}\left\{\frac{r}{pi+r}\binom{pi+r}{i}\right\}^{\alpha_i(P)} = \frac{1}{(m+1)}\frac{(m+j+1)r}{(m+j+1)r+mp}\binom{m+j}{m}\binom{r(m+j+1)+mp}{m}.$$

2.2. **A combinatorial proof of Corollary 2.3.** Let $\hat{\mathfrak{P}}_{n,k,r}^{m,j}$ be the set of lattice paths $P^* = Pd$ such that there is one colored down-step in P^*, where $P \in \mathfrak{P}_{n,k,r}^{m,j}$. To give a bijective proof of Corollary 2.3, we need the following lemma.

Lemma 2.15. *There exists a bijection ϕ between the sets $\hat{\mathfrak{P}}_{n,k,r}^{m,j}$ and $\mathfrak{L}_{n,k,r}^{m,j}$ such that $P^* \in \hat{\mathfrak{P}}_{n,k,r}^{m,j}$ has the same type of u-segments as $\phi(P^*) \in \mathfrak{L}_{n,k,r}^{m,j}$.*

Proof. Any $P^* \in \hat{\mathfrak{P}}_{n,k,r}^{m,j}$ can be uniquely partitioned into $P^* = P_1dQ_1$, where P_1, Q_1 are lattice paths and d is the colored down-step. Define $\phi(P^*) = Q_1P_1$, then it is easy to verify that $\phi(P^*) \in \mathfrak{L}_{n,k,r}^{m,j}$ and note that the length of any u-segment in $\phi(P^*)$ is the same as in P^*.

Conversely, for any $L \in \mathfrak{L}_{n,k,r}^{m,j}$, we can find the leftmost point which has the lowest ordinate, then L can be uniquely partitioned into two parts in this sense, namely, $L = L_1L_2$. Define $\phi^{-1}(L) = L_2dL_1$, where the d is the colored down-step, then it is easily to verify that $\phi^{-1}(L) \in \hat{\mathfrak{P}}_{n,k,r}^{m,j}$ which has the same type of u-segments with L.

Hence ϕ is indeed a bijection between the sets $\hat{\mathfrak{P}}_{n,k,r}^{m,j}$ and $\mathfrak{L}_{n,k,r}^{m,j}$, which preserves the type of u-segments not changed. □

An *ordered partition* $B_1, B_2, \cdots, B_r$ of $[m] = \{1, 2, \cdots, m\}$ into r blocks is a partition of $[m]$ such that the r blocks as well as the elements of each block are ordered.

Now we can give a bijective proof of Corollary 2.3.

Proof. For any ordered partition $B_1, B_2, \cdots, B_r$ of $[m]$ into r blocks, regard each block B_i as a labeled u-segment U_i for $1 \leq i \leq r$. For m down-steps and j horizontal-steps, there are $\binom{m+j}{j}$ ways to obtain (d, h)-words of length $m+j$ on $\{d, h\}$ with m d's and j h's. Then we can insert the labeled u-segments $U_1, U_2, \cdots, U_r$ orderly into the $m + j + 1$ positions (repetition is not allowed) of any (d, h)-word of length $m + j$, which can produce $\binom{m+j+1}{r}$ labeled lattice paths in $\mathfrak{L}_{n,k,r}^{m,j}$. Note that $r!\mathbf{B}_{m,r}(1!t_1, 2!t_2, \cdots)$ is just the generating function for ordered partitions $B_1, B_2, \cdots, B_r$ of $[m]$ into r blocks such that each block B_p is weighted by t_i with $i = |B_p|$ for $1 \leq p \leq r$. So $\binom{m+j}{m}\binom{m+j+1}{r}r!\mathbf{B}_{m,r}(1!t_1, 2!t_2, \cdots)$ is the generating function for the labeled lattice paths in $\mathfrak{L}_{n,k,r}^{m,j}$ such that each u-segment of length i is weighted by t_i.

However, by Lemma 2.15, any k-path $P \in \mathfrak{P}_{n,k,r}^{m,j}$ can lead to $m!$ labeled k-paths, and $P^* = Pd \in \hat{\mathfrak{P}}_{n,k,r}^{m,j}$ can generate $m + 1$ lattice paths in $\mathfrak{L}_{n,k,r}^{m,j}$ and vice versa. Hence $\frac{1}{m+1}\binom{m+j}{m}\binom{m+j+1}{r}\frac{r!}{m!}\mathbf{B}_{m,r}(1!t_1, 2!t_2, \cdots)$ is the generating function of k-paths in $\mathfrak{P}_{n,k,r}^{m,j}$ such that each u-segment of length i is weighted by t_i, which makes the proof complete. □

3. "INTERNAL u-SEGMENTS" STATISTICS IN k-PATHS

An *internal u-segment* of a k-path P is a u-segment between two steps such as dd, hh, hd, dh, i.e., all u-segments except for the first one are internal u-segments. Define $\beta_r(P)$ to be the number internal u-segments of length r in a k-path P. We start this section by studying

the ordinary generating functions for the number of k-paths of length n according to the statistics $\beta_1, \beta_2, \ldots$, that is,

$$F(x,z;\mathbf{t}) = F(x,z;t_1,t_2,\ldots) = \sum_{m,j\geq 0} x^m z^j \sum_{P\in\mathfrak{P}_{n,k}^{m,j}} \prod_{i\geq 1} t_i^{\beta_i(P)},$$

which can be represented as follows in terms of the generating function $P(x,z;\mathbf{t})$.

Proposition 3.1. *The ordinary generating function $F(x,z;\mathbf{t})$ is given by*

$$\frac{1+zP(x,z;\mathbf{t})}{1-xP(x,z;\mathbf{t})} = 1 + zP(x,z;\mathbf{t}) + \sum_{j\geq 1} x^j P^j(x,z;\mathbf{t}) + z\sum_{j\geq 1} x^j P^{j+1}(x,z;\mathbf{t}).$$

Proof. An equation for $F(x,z;\mathbf{t})$ is obtained from the decomposition of a k-path: either

$$P = hP',\ P = u^j dP^{(j)} dP^{(j-1)} \cdots dP^{(2)} dP^{(1)},\ \text{or}\ P = u^j hP^{(j+1)} dP^{(j)} \cdots dP^{(2)} dP^{(1)}$$

for some $j \geq 1$, where $P', P^{(1)}, \cdots, P^{(j+1)}$ are k-paths. Then $F(x,z;\mathbf{t})$ satisfies the equation $F(x,z;\mathbf{t}) = 1 + zP(x,z;\mathbf{t}) + \sum_{j\geq 1} x^j P^j(x,z;\mathbf{t}) + z\sum_{j\geq 1} x^j P^{j+1}(x,z;\mathbf{t})$, as required. □

Theorem 3.2. *For any integers $k, j \geq 0$, $n, m \geq 1$,*

$$\begin{aligned}
\sum_{P\in\mathfrak{P}_{n,k}^{m,j}} \prod_{i\geq 1} t_i^{\beta_i(P)} &= \sum_{p=0}^{m} \left\{ \frac{m-p}{m+j}\binom{m+j}{j} + \binom{m+j}{j-1}\frac{m-p+1}{m+j} \right\} \mathbf{P}_p^{(m+j)}(1!t_1, 2!t_2, \cdots) \\
&= \binom{m+j}{j} \sum_{p=0}^{m} \frac{(m+1)(m-p)+j(m-p+1)}{(m+1)(m+j)} \sum_{r=0}^{p} \binom{m+j}{r} \frac{r!}{p!} \mathbf{B}_{p,r}(1!t_1, 2!t_2, \cdots).
\end{aligned}$$

Proof. Using (1.2) and (1.3), we obtain

$$\begin{aligned}
\sum_{P\in\mathfrak{P}_{n,k}^{m,j}} \prod_{i\geq 1} t_i^{\beta_i(P)} &= [x^m z^j]\frac{1+zP(x,z;\mathbf{t})}{1-xP(x,z;\mathbf{t})} = [x^{m+j}z^j]\frac{1+zy^*}{1-y^*} \\
&= \frac{1}{m+j}[x^{m+j-1}z^j]\left\{\frac{1+zx}{1-x}\right\}' \big((1+zx)T(x)\big)^{m+j} \\
&= \frac{1}{m+j}\binom{m+j}{j}[x^{m-1}]\frac{T(x)^{m+j}}{(1-x)^2} + \frac{1}{m+j}\binom{m+j}{j-1}[x^m]\frac{T(x)^{m+j}}{(1-x)^2} \\
&= \binom{m+j}{j}\sum_{p=0}^{m}\frac{m-p}{m+j}\mathbf{P}_p^{(m+j)}(1!t_1, 2!t_2, \cdots) + \binom{m+j}{j-1}\sum_{p=0}^{m}\frac{m-r+1}{m+j}\mathbf{P}_p^{(m+j)}(1!t_1, 2!t_2, \cdots) \\
&= \sum_{p=0}^{m} \left\{ \frac{m-p}{m+j}\binom{m+j}{j} + \binom{m+j}{j-1}\frac{m-p+1}{m+j} \right\} \mathbf{P}_p^{(m+j)}(1!t_1, 2!t_2, \cdots) \\
&= \binom{m+j}{j} \sum_{p=0}^{m} \frac{(m+1)(m-p)+j(m-p+1)}{(m+1)(m+j)} \sum_{r=0}^{p} \binom{m+j}{r} \frac{r!}{p!} \mathbf{B}_{p,r}(1!t_1, 2!t_2, \cdots),
\end{aligned}$$

as claimed. □

4. "u-segments" and "Internal u-segments" Statistics in k-paths without a horizontal-step on the x-axis

4.1. **u-segments statistics.** We start this subsection by studying the generating function for the number of k-paths in $\mathfrak{Q}_{n,k}^{m,j}$ according to the statistics $\alpha_1, \alpha_2, \ldots$, that is,

$$Q(x,z;\mathbf{t}) = Q(x,z;t_1,t_2,\ldots) = \sum_{m,j\geq 0} x^m z^j \sum_{Q\in\mathfrak{Q}_{n,k}^{m,j},} \prod_{i\geq 1} t_i^{\alpha_i(Q)}.$$

Proposition 4.1. *The ordinary generating function $Q(x,z;\mathbf{t})$ is given by*

$$Q(x,z;\mathbf{t}) = \frac{P(x,z;\mathbf{t})}{1+zP(x,z;\mathbf{t})} = T(y). \tag{4.1}$$

Proof. Note that $Q(x,z;\mathbf{t})$ can be written as $Q(x,z;\mathbf{t}) = 1+\sum_{j\geq 1} Q_p(x,z;\mathbf{t})$, where $Q_p(x,z;\mathbf{t})$ is the generating function for the number of k-paths starting with p up-steps and without a horizontal-step on the x-axis according to the statistics $\alpha_1, \alpha_2, \ldots$. An equation for $Q_p(x,z;\mathbf{t})$ is obtained from the first return decomposition of a k-path starting with a u-segment of length p: either $P = u^p dP^{(p-1)}dP^{(p-2)}d\ldots P^{(1)}dP^*$ or $P = u^p hP^{(p)}dP^{(p-1)}d\ldots P^{(1)}dP^*$, where $P^{(1)},\ldots,P^{(p)}$ are k-paths and P^* is a k-path without a horizontal-step on the x-axis. Thus $Q_p(x,z;\mathbf{t}) = t_p x^p P^{p-1}(x,z;\mathbf{t})Q(x,z;\mathbf{t}) + zt_p x^p P^p(x,z;\mathbf{t})Q(x,z;\mathbf{t})$ and $Q(x,z;\mathbf{t})$ satisfies the equation $Q(x,z;\mathbf{t}) = 1+Q(x,z;\mathbf{t})\left\{\sum_{p\geq 1} t_p x^p P^{p-1}(x,z;\mathbf{t}) + z\sum_{p\geq 1} t_p x^p P^p(x,z;\mathbf{t})\right\}$. Hence, by Proposition 2.1, we obtain the desired result. □

Theorem 4.2. *For any integers $k,j\geq 0$, $n,m\geq 1$,*

$$\begin{aligned}
\sum_{Q\in\mathfrak{Q}_{n,k}^{m,j}} \prod_{i\geq 1} t_i^{\alpha_i(Q)} &= \frac{m}{(m+j+1)(m+j)}\binom{m+j}{j}\frac{\mathbf{P}_m^{(m+j+1)}(1!t_1,2!t_2,\cdots)}{m!}\\
&= \frac{m}{(m+j+1)(m+j)}\binom{m+j}{j}\sum_{r=0}^{m}\binom{m+j+1}{r}\frac{r!}{m!}\mathbf{B}_{m,r}(1!t_1,2!t_2,\cdots).
\end{aligned}$$

Proof. Using (1.2) and (1.3), we obtain

$$\begin{aligned}
\sum_{Q\in\mathfrak{Q}_{n,k}^{m,j}} \prod_{i\geq 1} t_i^{\alpha_i(Q)} &= [x^m z^j]Q(x,z;\mathbf{t}) = [x^{m+j}z^j]T(y^*)\\
&= \frac{1}{m+j}[x^{m+j-1}z^j]\left\{T(x)\right\}'\left\{(1+zx)T(x)\right\}^{m+j}\\
&= \frac{1}{m+j}\binom{m+j}{j}[x^{m-1}]\left\{T(x)\right\}'\left\{T(x)\right\}^{m+j}\\
&= \frac{1}{m+j}\binom{m+j}{j}\frac{1}{m+j+1}[x^{m-1}]\left\{T(x)^{m+j+1}\right\}'\\
&= \frac{m}{(m+j+1)(m+j)}\binom{m+j}{j}[x^m]T(x)^{m+j+1}\\
&= \frac{m}{(m+j+1)(m+j)}\binom{m+j}{j}\frac{\mathbf{P}_m^{(m+j+1)}(1!t_1,2!t_2,\cdots)}{m!}\\
&= \frac{m}{(m+j+1)(m+j)}\binom{m+j}{j}\sum_{r=0}^{m}\binom{m+j+1}{r}\frac{r!}{m!}\mathbf{B}_{m,r}(1!t_1,2!t_2,\cdots),
\end{aligned}$$

which completes the proof. □

Replacing t_i by qt_i in Theorem 4.2 and using (2.2) and (2.3), we have

Corollary 4.3. *For any integers $n, m, r \geq 1$ and $k, j \geq 0$, there holds*

$$\sum_{Q\in\mathfrak{Q}_{n,k,r}^{m,j}}\prod_{i\geq 1} t_i^{\alpha_i(Q)} = \frac{m}{(m+j+1)(m+j)}\binom{m+j}{j}\binom{m+j+1}{r}\frac{r!}{m!}\mathbf{B}_{m,r}(1!t_1, 2!t_2, \cdots),$$

which implies that the number of k-paths with no horizontal-step on the x-axis and with u-segments of type $1^{r_1}2^{r_2}\cdots m^{r_m}$ is $\frac{m}{(m+j+1)(m+j)}\binom{m+j}{m}\binom{m+j+1}{r}\binom{r}{r_1,r_2,\cdots,r_m}$.

Remark 4.4. *Note that from Theorem 2.2 and 4.2, the ratio of $\sum_{P\in\mathfrak{P}_{n,k}^{m,j}}\prod_{i\geq 1} t_i^{\alpha_i(P)}$ to $\sum_{Q\in\mathfrak{Q}_{n,k}^{m,j}}\prod_{i\geq 1} t_i^{\alpha_i(Q)}$ is $\frac{(m+1)m}{(m+j+1)(m+j)}$. For the sake of conciseness, we omit many examples as done in Section 2. However, we may ask whether there is a combinatorial interpretation for this relation.*

4.2. **Internal u-segments statistics.** In this subsection, we study the generating function for the number of k-paths in $\mathfrak{Q}_{n,k}^{m,j}$ according to the statistics $\beta_1, \beta_2, \ldots$, that is,

$$H(x,z;\mathbf{t}) = H(x,z;t_1,t_2,\ldots) = \sum_{m,j\geq 0} x^m z^j \sum_{Q\in\mathfrak{Q}_{n,k}^{m,j},}\prod_{i\geq 1} t_i^{\beta_i(Q)}.$$

Proposition 4.5. *The ordinary generating function $H(x,z;\mathbf{t})$ is given by*

$$H(x,z;\mathbf{t}) = \frac{1-xP(x,z;\mathbf{t})}{1-x-xP(x,z;\mathbf{t})-zxP(x,z;\mathbf{t})} = \frac{1-y}{1-y-x-zy}. \tag{4.2}$$

Proof. Note that $H(x,z;\mathbf{t})$ can be written as $H(x,z;\mathbf{t}) = 1+\sum_{p\geq 1} H_p(x,z;\mathbf{t})$, where $H_p(x,z;\mathbf{t})$ is the generating function for the number of k-paths starting with p up-steps and without a horizontal-step on the x-axis according to the statistics $\beta_1, \beta_2, \ldots$. An equation for $H_p(x,z;\mathbf{t})$ is obtained from the first return decomposition of a k-path starting with a u-segment of length p: either $P = u^p dP^{(p-1)}dP^{(p-2)}d\ldots P^{(1)}dP^*$ or $P = u^p hP^{(p)}dP^{(p-1)}d\ldots P^1 dP^*$, where $P^{(1)}, \ldots, P^{(p)}$ are k-paths and P^* is a k-path with no horizontal-step on the x-axis. Thus $H_p(x,z;\mathbf{t}) = x^p P^{p-1}(x,z;\mathbf{t})H(x,z;\mathbf{t}) + zx^p P^p(x,z;\mathbf{t})H(x,z;\mathbf{t})$. Hence, $H(x,z;\mathbf{t})$ satisfies the equation $H(x,z;\mathbf{t}) = 1 + \left\{\sum_{p\geq 1} x^p P^{p-1}(x,z;\mathbf{t}) + z\sum_{p\geq 1} x^p P^p(x,z;\mathbf{t})\right\}H(x,z;\mathbf{t})$, a simplification reduces this to the required expression. □

Theorem 4.6. *For any integers $n, m, k, j \geq 0$, $m+j \geq 1$,*

$$\begin{aligned}\sum_{Q\in\mathfrak{Q}_{n,k}^{m,j}}\prod_{i\geq 1} t_i^{\beta_i(Q)} &= \sum_{i=0}^{m}\frac{i}{m+j-i}\binom{m+j-1}{j}\sum_{r=0}^{m-i-1}\binom{r+i}{r}\frac{\boldsymbol{P}_{m-i-r-1}^{(m+j-i)}(1!t_1, 2!t_2, \cdots)}{(m-i-r-1)!} \\ &+ \sum_{i=0}^{m}\frac{i}{m+j-i}\binom{m+j-1}{j-1}\sum_{r=0}^{m-i}\binom{r+i}{r}\frac{\boldsymbol{P}_{m-i-r}^{(m+j-i)}(1!t_1, 2!t_2, \cdots)}{(m-i-r)!}.\end{aligned}$$

Proof. Using (1.2) and (1.3), we obtain

$$\sum_{Q\in\mathfrak{Q}_{n,k}^{m,j}}\prod_{i\geq 1}t_i^{\beta_i(Q)} = [x^m z^j]Q(x,z;\mathbf{t}) = [x^{m+j}z^j]\frac{1-y^*}{1-y^*-x(1+zy^*)}$$

$$= [x^{m+j}z^j]\sum_{i\geq 0}^{m+j}\left\{\frac{x(1+zy^*)}{1-y^*}\right\}^i = \sum_{i\geq 0}^{m+j}[x^{m+j-i}z^j]\left\{\frac{1+zy^*}{1-y^*}\right\}^i$$

$$= \sum_{i\geq 0}^{m+j}\frac{i}{m+j-i}[x^{m+j-i-1}z^j]\left\{\frac{1+zx}{1-x}\right\}^{i-1}\left\{\frac{1+zx}{1-x}\right\}'\{(1+zx)T(x)\}^{m+j-i}$$

$$= \sum_{i\geq 0}^{m+j}\frac{i}{m+j-i}[x^{m+j-i-1}z^j]\left\{\frac{(1+z)(1+zx)^{m+j-1}}{(1-x)^{i+1}}\right\}\{T(x)\}^{m+j-i}$$

$$= \sum_{i\geq 0}^{m+j}\frac{i}{m+j-i}\left\{\binom{m+j-1}{j}[x^{m-i-1}]\frac{T(x)^{m+j-i}}{(1-x)^{i+1}}+\binom{m+j-1}{j-1}[x^{m-i}]\frac{T(x)^{m+j-i}}{(1-x)^{i+1}}\right\}$$

$$= \sum_{i=0}^{m}\frac{i}{m+j-i}\binom{m+j-1}{j}\sum_{r=0}^{m-i-1}\binom{r+i}{r}\frac{\mathbf{P}_{m-i-r-1}^{(m+j-i)}(1!t_1,2!t_2,\cdots)}{(m-i-r-1)!}$$

$$+\sum_{i=0}^{m}\frac{i}{m+j-i}\binom{m+j-1}{j-1}\sum_{r=0}^{m-i}\binom{r+i}{r}\frac{\mathbf{P}_{m-i-r}^{(m+j-i)}(1!t_1,2!t_2,\cdots)}{(m-i-r)!},$$

which completes the proof. □

5. Statistics (u,h)-segments and u-segments in k-paths

A (u,h)-*segment* in a k-path is a maximum segment composed of up-steps and horizontal-steps. An *internal* (u,h)-*segment* in a k-path is a (u,h)-segment between two down steps. Let $\tilde{\mathfrak{P}}_{n,k}^{m,j,\ell}$ denote the subset of $\mathfrak{P}_{n,k}^{m,j}$ such that (i) each internal (u,h)-segment has length equal to a multiple of k; (ii) the first (u,h)-segment has length $\equiv \ell$ $(mod\ k)$ for $0\leq\ell\leq k-1$. We note that the case $j=0$ is studied in [9].

Theorem 5.1. *The number $\tilde{\mathcal{P}}_{r,k,\ell}$ of k-paths of length $n=kr+2\ell$ satisfying the conditions* (i) *and* (ii) *is*

$$\tilde{\mathcal{P}}_{r,k,\ell} = \sum_{p=0}^{[r/2]}\frac{\ell+1}{kp+\ell+1}\binom{(k+1)p+\ell}{p}\binom{r+2(k-1)p+2\ell}{r-2p}.$$

Proof. Note that for any path $P\in\tilde{\mathfrak{P}}_{n,k}^{m,j,\ell}$, by deleting all the j horizontal-steps, we obtain a Dyck path in $\tilde{\mathfrak{P}}_{2m,k}^{m,0,\ell}$. Conversely, any Dyck path in $\tilde{\mathfrak{P}}_{2m,k}^{m,0,\ell}$ goes through $2m+1$ integer points, if we insert the j horizontal-steps into any integer point (repetitions are allowed), then we get $\binom{2m+j}{j}$ k-paths in $\tilde{\mathfrak{P}}_{n,k}^{m,j,\ell}$. However, the set $\tilde{\mathfrak{P}}_{2m,k}^{m,0,\ell}$ is counted by $\frac{\ell+1}{m+1}\binom{m+p}{p}$, which has

been proved in [9]. Hence we have

$$\begin{aligned}
\tilde{\mathcal{P}}_{r,k,\ell} &= \sum_{2m+kj=n, m=kp+\ell} |\tilde{\mathfrak{P}}^{m,j,\ell}_{n,k}| \\
&= \sum_{2m+kj=n, m=kp+\ell} \frac{\ell+1}{m+1}\binom{m+p}{p}\binom{2m+j}{j} \\
&= \sum_{p=0}^{[r/2]} \frac{\ell+1}{kp+\ell+1}\binom{(k+1)p+\ell}{p}\binom{r+2(k-1)p+2\ell}{r-2p},
\end{aligned}$$

as required. □

Let $\bar{\mathfrak{P}}^{m,j,\ell}_{n,k}$ denote the subset of $\mathfrak{P}^{m,j}_{n,k}$ such that (iii) each internal u-segment has length equal to a multiple of k; (iv) the first u-segment has length $\equiv \ell$ $(mod\ k)$ for $0 \le \ell \le k-1$.

Theorem 5.2. *The number $\bar{\mathcal{P}}_{r,k,\ell}$ of k-paths of length $n = kr + 2\ell$ satisfying conditions* (iii) *and* (iv) *is*

$$\bar{\mathcal{P}}_{r,k,\ell} = \sum_{p=0}^{[r/2]} \frac{\ell+1}{kp+\ell+1}\binom{(k+1)p+\ell}{p}\binom{r+(k-1)p+\ell}{r-2p}.$$

Proof. Note that for any path $P \in \bar{\mathfrak{P}}^{m,j,\ell}_{n,k}$, by deleting all the j horizontal-steps, we obtain a Dyck path in $\bar{\mathfrak{P}}^{m,0,\ell}_{2m,k}$. Conversely,let D be a Dyck path from $\bar{\mathfrak{P}}^{m,0,\ell}_{2m,k}$, where $m = kp+\ell$ for some $p \ge 0$. It can be shown that D has $m+p+1$ proper integer points, where a proper integer point is a point where we may insert a horizontal step without violating the properties (iii) and (iv). By inserting j horizontal steps into these points (repetitions are allowed) we get $\binom{m+p+j}{j}$ k-paths in $\bar{\mathfrak{P}}^{m,j,\ell}_{2m,k}$. Note that the set $\bar{\mathfrak{P}}^{m,0,\ell}_{2m,k} = \tilde{\mathfrak{P}}^{m,0,\ell}_{2m,k}$ is counted by $\frac{\ell+1}{m+1}\binom{m+p}{p}$. Hence we have

$$\begin{aligned}
\bar{\mathcal{P}}_{r,k,\ell} &= \sum_{2m+kj=n, m=kp+\ell} |\bar{\mathfrak{P}}^{m,j,\ell}_{n,k}| \\
&= \sum_{2m+kj=n, m=kp+\ell} \frac{\ell+1}{m+1}\binom{m+p}{p}\binom{m+p+j}{j} \\
&= \sum_{p=0}^{[r/2]} \frac{\ell+1}{kp+\ell+1}\binom{(k+1)p+\ell}{p}\binom{r+(k-1)p+\ell}{r-2p},
\end{aligned}$$

as required. □

It should be noted that $\sum_{p\ge0} \frac{\ell+1}{kp+\ell+1}\binom{(k+1)p+\ell}{p}x^p = f(x)^{\ell+1}$, where $f(x)$ is the generating function for $(k+1)$-ary plane trees, and which satisfies the relation $f(x) = 1 + xf(x)^{k+1}$. Then it is easy to prove that the generating functions for $\tilde{\mathcal{P}}_{r,k,\ell}$ and for $\bar{\mathcal{P}}_{r,k,\ell}$ are respectively

$$\tilde{P}_{k,\ell}(x) = \sum_{r\ge0} \tilde{\mathcal{P}}_{r,k,\ell}x^r = \frac{1}{(1-x)^{2\ell+1}} f(\frac{x^2}{(1-x)^{2k}})^{\ell+1}, \tag{5.1}$$

$$\bar{P}_{k,\ell}(x) = \sum_{r\ge0} \bar{\mathcal{P}}_{r,k,\ell}x^r = \frac{1}{(1-x)^{\ell+1}} f(\frac{x^2}{(1-x)^{k+1}})^{\ell+1}. \tag{5.2}$$

Replacing x by $\frac{x}{1+x}$ in (5.1) and (5.2), one can deduce that

$$\frac{1}{(1+x)^{2\ell+1}}\tilde{P}_{k,\ell}(\frac{x}{1+x}) = f(x^2(1+x)^{2k-2})^{\ell+1}, \tag{5.3}$$

$$\frac{1}{(1+x)^{\ell+1}}\bar{P}_{k,\ell}(\frac{x}{1+x}) = f(x^2(1+x)^{k-1})^{\ell+1}. \tag{5.4}$$

Comparing the coefficient of x^n in both sides of (5.3) and (5.4), one can deduce the following consequence:

Corollary 5.3. *For any integers $n, k, \ell \geq 0$, there hold*

$$\sum_{p=0}^{n}(-1)^{n-p}\binom{n+2\ell}{p+2\ell}\tilde{\mathcal{P}}_{p,k,\ell} = \sum_{p=0}^{[\frac{n}{2}]}\frac{\ell+1}{kp+\ell+1}\binom{(k+1)p+\ell}{p}\binom{2(k-1)p}{n-2p},$$

$$\sum_{p=0}^{n}(-1)^{n-p}\binom{n+\ell}{p+\ell}\bar{\mathcal{P}}_{p,k,\ell} = \sum_{p=0}^{[\frac{n}{2}]}\frac{\ell+1}{kp+\ell+1}\binom{(k+1)p+\ell}{p}\binom{(k-1)p}{n-2p}. \tag{5.5}$$

Using the generalized Lagrange inversion formula obtained in [6], from (5.1), we have

$$\begin{aligned}
&\frac{\ell+1}{kn+\ell+1}\binom{(k+1)n+\ell}{n} = [x^n]f(x)^{\ell+1}\\
&= [w^n]\left\{(1-x)^{2\ell+1}\tilde{P}_{k,\ell}(x)\right\}_{w=\frac{x^2}{(1-x)^{2k}}}\\
&= \frac{1}{2n}[t^{2n-1}](1-t)^{2kn}\frac{d}{dt}\left\{(1-t)^{2\ell+1}\tilde{P}_{k,\ell}(t)\right\}\\
&= \frac{2\ell+1}{2n}[t^{2n-1}](1-t)^{2kn+2\ell}\tilde{P}_{k,\ell}(t) + \frac{1}{2n}[t^{2n-1}](1-t)^{2kn+2\ell+1}\frac{d}{dt}\tilde{P}_{k,\ell}(t)\\
&= \sum_{p=0}^{2n-1}(-1)^p\binom{2kn+2\ell}{2n-p-1}\frac{2\ell+1}{2n}\tilde{\mathcal{P}}_{p,k,\ell} + \sum_{p=1}^{2n}(-1)^p\binom{2kn+2\ell+1}{2n-p}\frac{p}{2n}\tilde{\mathcal{P}}_{p,k,\ell}\\
&= \sum_{p=0}^{2n}(-1)^p\frac{kp+2\ell+1}{2kn+2\ell+1}\binom{2kn+2\ell+1}{2n-p}\tilde{\mathcal{P}}_{p,k,\ell}.
\end{aligned}$$

Similarly, from (5.2), we have

$$\begin{aligned}
&\frac{\ell+1}{kn+\ell+1}\binom{(k+1)n+\ell}{n} = [x^n]f(x)^{\ell+1}\\
&= [w^n]\left\{(1-x)^{\ell+1}\bar{P}_{k,\ell}(x)\right\}_{w=\frac{x^2}{(1-x)^{k+1}}}\\
&= \frac{1}{2n}[t^{2n-1}](1-t)^{(k+1)n}\frac{d}{dt}\left\{(1-t)^{\ell+1}\bar{P}_{k,\ell}(t)\right\}\\
&= \sum_{p=0}^{2n}(-1)^p\frac{p(k+1)+2(\ell+1)}{2n(k+1)+2(\ell+1)}\binom{n(k+1)+\ell+1}{2n-p}\bar{\mathcal{P}}_{p,k,\ell}.
\end{aligned}$$

Hence we obtain the next corollary:

Corollary 5.4. *For any integers $n, k, \ell \geq 0$, it holds that*

$$(5.6)\quad \frac{\ell+1}{kn+\ell+1}\binom{(k+1)n+\ell}{n} = \sum_{p=0}^{2n}(-1)^p\frac{kp+2\ell+1}{2kn+2\ell+1}\binom{2kn+2\ell+1}{2n-p}\tilde{\mathcal{P}}_{p,k,\ell},$$

$$(5.7)\quad \frac{\ell+1}{kn+\ell+1}\binom{(k+1)n+\ell}{n} = \sum_{p=0}^{2n}(-1)^p\frac{p(k+1)+2(\ell+1)}{2n(k+1)+2(\ell+1)}\binom{n(k+1)+\ell+1}{2n-p}\bar{\mathcal{P}}_{p,k,\ell}.$$

We consider below several special cases, leading to several interesting results.

Example 5.5. *If $k=1$ and $\ell=0$ in (5.2), then $f(x)=\frac{1-\sqrt{1-4x}}{2x}=C(x)$, which is the generating function for the Catalan numbers $C_n=\frac{1}{n+1}\binom{2n}{n}$. Hence we have*

$$\bar{P}_{1,0}(x)=\frac{1}{1-x}C(\frac{x^2}{(1-x)^2})=\frac{1-x-\sqrt{1-2x-3x^2}}{2x^2},$$

which is the generating function $M(x)$ for the Motzkin numbers M_n. Then Theorem 5.2 together with (5.5) and (5.6) generates the well-known identities (see [1, 3]*)*

$$\sum_{p=0}^{[n/2]}\binom{n}{2p}C_p = M_n,$$

$$\sum_{p=0}^{n}(-1)^{n-p}\binom{n}{p}M_p = \begin{cases} C_r & \text{if } n=2r,\\ 0 & \text{otherwise.}\end{cases}$$

Example 5.6. *If $k=2$ and $\ell=0$ in (5.2), then $f(x)=1+xf(x)^3=\sum_{n\geq 0}\frac{1}{2n+1}\binom{3n}{n}x^n$, which is the generating function for complete 3-ary plane trees. So we have*

$$\bar{P}_{2,0}(x)=\frac{1}{1-x}f(\frac{x^2}{(1-x)^3})=\frac{1}{1-x}+\frac{x^2}{(1-x)^4}f(\frac{x^2}{(1-x)^3})^3=\frac{1}{1-x}(1+x^2\bar{P}_{2,0}(x)^3).$$

If we let $y=x\bar{P}_{2,0}(x)$, it follows that $y=x(1+y)^{-1}$. From this,

$$\bar{P}_{2,0}(x)=C(x)=\frac{1-\sqrt{1-4x}}{2x}.$$

Similarly, when $k=2$ and $\ell=1$ in (5.2), we have

$$\bar{P}_{2,1}(x)=\bar{P}_{2,0}(x)^2=C(x)^2=\frac{1-2x-\sqrt{1-4x}}{2x^2}=\frac{C(x)-1}{x}.$$

Hence we obtain the following statement:

Corollary 5.7. *The number of 2-paths (i.e. Schröder paths) of length $2n$ such that all u-segments have even length is the Catalan number C_n for $n\geq 0$ and the number of 2-paths of length $2n+2$ such that all internal u-segments have even length and the first u-segment has odd length is the Catalan number C_{n+1} for $n\geq 0$.*

Here is a simple bijective proof. For any Schröder path S of length $2n$ such that all u-segments have even length, replace each h step by ud steps, then we get a Dyck path of length $2n$. On the other hand, any Dyck path D of length $2n$ can be decomposed uniquely into $D=u^{i_1}d^{j_1}u^{i_2}d^{j_2}\cdots u^{i_k}d^{j_k}$, where $i's, j's\geq 1$. Now replace a sub-path $u^{i_l}d^{j_l}$ by $u^{i_l-1}hd^{j_l-1}$ if i_l is odd, and do nothing is i_l is even. Then we get a desired Schröder path S. A similar argument proves the second claim in Corollary 5.7.

Theorem 5.2 and Example 5.6 give rise to two new expressions for the Catalan numbers,

Corollary 5.8. *For any integer $n \geq 0$,*

$$\sum_{p=0}^{[n/2]} \frac{1}{2p+1}\binom{3p}{p}\binom{n+p}{3p} = \frac{1}{n+1}\binom{2n}{n}, \quad (n \geq 0),$$

$$\sum_{p=0}^{[(n-1)/2]} \frac{1}{2p+1}\binom{3p+1}{p+1}\binom{n+p}{3p+1} = \frac{1}{n+1}\binom{2n}{n}, \quad (n \geq 1).$$

Example 5.6 together with (5.5) and (5.6) leads to several new identities involving Catalan numbers

Corollary 5.9. *For any integer $n \geq 0$,*

$$\sum_{p=0}^{n}(-1)^{n-p}\binom{n}{p}C_p = \sum_{p=0}^{[\frac{n}{2}]} \frac{1}{2p+1}\binom{3p}{p}\binom{p}{n-2p}, \tag{5.8}$$

$$\sum_{p=0}^{n}(-1)^{n-p}\binom{n+1}{p+1}C_{p+1} = \sum_{p=0}^{[\frac{n}{2}]} \frac{1}{2p+1}\binom{3p+1}{p+1}\binom{p}{n-2p},$$

$$\sum_{p=0}^{2n}(-1)^{p}\frac{3p+2}{2n+2p+2}\binom{3n}{n+p}C_p = \frac{1}{2n+1}\binom{3n}{n},$$

$$\sum_{p=0}^{2n}(-1)^{p}\frac{3p+4}{2n+2p+4}\binom{3n+1}{n+p+1}C_{p+1} = \frac{1}{2n+1}\binom{3n+1}{n+1}.$$

Remark 5.10. (I) *In fact, the counting formula in Theorem 5.1 and (5.6), and the counting formula in Theorem 5.2 and (5.7) form two left-inversion relations which have the general formats obtained implicitly by Corsani, Merlini and Sprugnoli* [6], *namely*

$$A_n = \sum_{p=0}^{[\frac{n}{2}]}\binom{n+2(k-1)p+2\ell}{n-2p}B_p \Rightarrow B_n = \sum_{p=0}^{2n}(-1)^p\frac{kp+2\ell+1}{2kn+2\ell+1}\binom{2kn+2\ell+1}{2n-p}A_p,$$

$$A_n = \sum_{p=0}^{[\frac{n}{2}]}\binom{n+(k-1)p+\ell}{n-2p}B_p \Rightarrow B_n = \sum_{p=0}^{2n}(-1)^p\frac{p(k+1)+2(\ell+1)}{2n(k+1)+2(\ell+1)}\binom{n(k+1)+\ell+1}{2n-p}A_p.$$

(II) *It should be noticed that the left side of (5.8) is the Riordan numbers obtained by Bernhart* [3]*, so the right side of (5.8) gives a new expression for the Riordan numbers.*

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